

Local Rigidity of Diophantine translations in higher dimensional tori

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Abstract

We prove a theorem asserting that, given a Diophantine rotation α in a torus $\mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$, any perturbation, small enough in the C^∞ topology, that does not destroy all orbits with rotation vector α is actually smoothly conjugate to the rigid rotation. The proof relies on a K.A.M. scheme (named after Kolmogorov-Arnol'd-Moser), where at each step the existence of an invariant measure with rotation vector α assures that we can linearize the equations around the same rotation α . The proof of the convergence of the scheme is carried out in the C^∞ category.

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1 Introduction

Let $R_\alpha : x \mapsto x + \alpha \pmod{\mathbb{Z}^d}$ be a translation on a torus \mathbb{T}^d with $d \in \mathbb{N}^*$. The search for conditions under which a diffeomorphism $f \in \text{Diff}^\infty(\mathbb{T}^d)$ is guaranteed to be smoothly conjugate to R_α is a very old subject in dynamical systems and the source of very deep and far-reaching studies, see for example [Her79] and [Yoc95] for the case $d = 1$.

To our best knowledge, the strongest rigidity result on perturbations of Diophantine rotations in higher dimensional tori in the literature is the one proved in [Her79]. This theorem, apart from the smallness assumptions, needs the preservation of a volume form, something that assures that *every* orbit rotates at the speed of the Diophantine rotation, so that the analogy with the one-dimensional theory is direct.

Our goal in the present article is to relax the condition of preservation of a (harmonic) volume form to a considerably weaker one, which seems partly optimal. The closeness-to-rotations condition is, a priori at least, not indispensable, while the Diophantine property is known to be thus, since in the Liouvillean world rigid rotations tend to be fragile. The present rigidity theorem, whose precise statement is given in thm 3.1, is in fact an instance of the strength of the Diophantine condition and of the K.A.M. machinery.

Theorem A. *Let $\alpha \in \mathbb{T}^d$, $d \in \mathbb{N}^*$, be a Diophantine rotation and $f \in \text{Diff}^\infty(\mathbb{T}^d)$ be a small enough perturbation. Then, if α is in the convex hull of the rotation set of f , the diffeomorphism f is smoothly conjugate to the translation by α .*

The motivation for this theorem comes from a conjecture concerning diffeomorphisms of tori of dimension higher than 1. In the one-dimensional case, the celebrated Denjoy theorem and examples establish a break in dynamical behaviour at the regularity threshold $C^{1+\text{BV}}(\mathbb{T}^1)$.¹ A circle diffeomorphism with irrational rotation number and regularity lower than $C^{1+\text{BV}}$ may have wandering intervals, while a diffeomorphism of regularity $C^{1+\text{BV}}$ cannot (see, e.g. [KH96]).

In the one-dimensional case (see e.g. [dMvS93] or [Her79]), a homeomorphism is assigned a unique rotation number, and, as soon as it is irrational, a continuous semi-conjugation to the rigid rotation can be readily constructed. Denjoy's theorem is a rigidity theorem, stating that if the homeomorphism is sufficiently regular, the semi-conjugacy is in fact automatically a continuous conjugacy. Arnold's theorem and subsequently the Herman-Yoccoz theory ([Her79] and [Yoc95]) is a further rigidity result in this setting, under additional regularity and arithmetical assumptions.

It is not known whether Denjoy's theory admits a reasonable generalization when the dimension of the torus is higher than one, but it is certainly not directly generalizable (due to the fact that the Denjoy-Koksma inequality fails, see [Yoc95]). A homeomorphism of a higher dimensional torus does not, in

¹By $C^{1+\text{BV}}(\mathbb{T}^1)$ we denote the space of circle diffeomorphisms of the circle whose first derivative has bounded variation.

general, have a unique rotation vector (see again [Her79]), and even if this is the case, the minimality of the corresponding translation does not imply the existence of a continuous semi-conjugacy to it (e.g. [Fur61]).

It is conjectured in [McS93], however, that for diffeomorphisms f of \mathbb{T}^d , with $d \geq 2$, who satisfy the *additional* assumption that there exists $\phi : \mathbb{T}^d \rightarrow \mathbb{T}^d$, continuous and surjective and such that

$$\phi \circ f = R_\alpha \circ \phi$$

with R_α minimal, a similar break should appear. That is, it is possible to construct such diffeomorphisms with wandering domains as long as the diffeomorphism is in, say $C^{d+1-\varepsilon}$, but not if it is more regular than, say, C^{d+1} .

To our best knowledge, some examples of particular nature and of regularity lower than the conjectured threshold have been constructed by McSwiggen (see [McS93] and [McS95]) and by Sambarino and Passeggi (see [PS13]). McSwiggen's examples on \mathbb{T}^2 are based on a Derived Anosov technique. A linear Anosov diffeomorphism of \mathbb{T}^3 is deformed in C^∞ in order to turn the saddle around the fixed point into a repeller. A diffeomorphism of \mathbb{T}^2 is then constructed as the holonomy map along the unstable foliation that is proved to survive the deformation. The radical loss of regularity from C^∞ to $C^{3-\varepsilon}$ comes from an inequality that has to be satisfied by an algebraic function of the eigenvalues of the original Anosov system and the need for contraction in the functional space of some bundle sections, and therefore in a quite indirect manner. The diffeomorphism thus constructed is proved to have wandering domains, while it is semi-conjugate to the unstable holonomy map of the original Anosov diffeomorphism by collapsing the repelling basin of the origin to a point. Moreover, even though it is not actually mentioned in the paper, the rotation vector is in fact Diophantine, since both the direction and the modulus of the translation vector are given by algebraic functions.

We think that, when compared to our result, this construction represents an instance of the fact that in low regularity arithmetic properties are irrelevant, while they become crucially so above some finite (and hopefully universal and explicit) threshold.

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2 Notation

2.1 General notation

By $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ we denote a lift of f (and in general the corresponding capital letter will denote a lift whenever the small one denotes a diffeomorphism of the torus). By a tilde we denote the lift of a point $x \in \mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$ to a representative in the covering space $\tilde{x} \in \mathbb{R}^d$.

A special case of diffeomorphisms of the torus is that of translations. For $\alpha \in \mathbb{T}^d$, we define

$$R_\alpha : x \mapsto x + \alpha \pmod{\mathbb{Z}^d}$$

We will denote the space of C^s -smooth diffeomorphisms that are isotopic to the identity by $\text{Diff}_0^s(\mathbb{T}^d)$, and the distance in Diff_0^s between two diffeomorphisms f and g by

$$d_s(f, g) = \max_{0 \leq \sigma \leq s} \|D^\sigma F - D^\sigma G\|_{L^\infty}$$

The space of C^∞ diffeomorphisms will be furnished with the corresponding topology.

If $\varphi : \mathbb{T}^d \rightarrow \mathbb{R}$, $\hat{\varphi}(k), k \in \mathbb{Z}^d$ are its Fourier coefficients, and $N \in \mathbb{N}^*$, we denote by

$$\begin{aligned} T_N \varphi(\cdot) &= \sum_{|k| \leq N} \hat{\varphi}(k) e^{2i\pi \cdot} \\ \dot{T}_N \varphi(\cdot) &= \sum_{0 < |k| \leq N} \hat{\varphi}(k) e^{2i\pi \cdot} \\ R_N \varphi(\cdot) &= \sum_{|k| > N} \hat{\varphi}(k) e^{2i\pi \cdot} \end{aligned}$$

the inhomogeneous and homogeneous truncations and the rest, respectively, where \mathbb{Z}^d is equipped with the ℓ^1 norm. The estimates

$$\begin{aligned} \|T_N \varphi(\cdot)\|_s &\leq C_s N^{s+d/2} \|\varphi\|_0 \\ \|\dot{T}_N \varphi(\cdot)\|_s &\leq C_s N^{s+d/2} \|\varphi\|_0 \\ \|R_N \varphi(\cdot)\|_s &\leq C_{s,s'} N^{-s'+s+d} \|\varphi\|_{s'}, \end{aligned}$$

are well known, where $0 \leq s \leq s'$.

If $f, g, u \in \text{Diff}_0^\infty(\mathbb{T}^d)$, then, see [Kri99],

$$\|g \circ f\|_s \leq C_s \|g\|_s (1 + \|f\|_s) (1 + \|f\|_0)^s \quad (1)$$

$$\|g \circ (f + u) - \psi \circ f\|_s \leq C_s \|g\|_{s+1} (1 + \|f\|_0)^s (1 + \|f\|_s) \|u\|_s \quad (2)$$

Finally, the vector $\alpha \in \mathbb{T}^d$ is said to satisfy a Diophantine condition of type γ, τ , if the following holds:

$$\alpha \in DC(\gamma, \tau) \Leftrightarrow \forall k \in \mathbb{Z}^d \setminus \{0\}, |k \cdot \alpha|_{\mathbb{Z}^d} \geq \frac{\gamma^{-1}}{|k|^{-1}}$$

2.2 Rotation vectors and sets

For this paragraph, see [MZ91] or [Fra95]. If $f \in \text{Homeo}_0(\mathbb{T}^d) \equiv \text{Diff}_0^0(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, we define $\rho(x, f)$ as the following limit, provided that it exists:

$$\frac{F^n(\tilde{x}) - \tilde{x}}{n}$$

It is defined $\bmod \mathbb{Z}^d$, due to the arbitrary choice of a lift for f .

We also define $\rho(f)$, the rotation set of f , as the accumulation points of

$$\frac{F^{n_i}(\tilde{x}_i) - \tilde{x}_i}{n_i}$$

where $n_i \rightarrow \infty$ and the $x_i \in \mathbb{T}^d$. It can be shown that $\rho(f)$ is the convex hull of $\bigcup_{x \in \mathbb{T}^d} \rho(x, f)$.

If $\nu \in \mathcal{M}(f)$ (i.e. a probability measure on \mathbb{T}^d , invariant under f) then the quantity

$$\int_{\mathbb{T}^d} (F(\tilde{x}) - \tilde{x}) d\nu$$

is well defined and denoted by $\rho(\nu, f)$. The set $\bigcup \rho(\nu, f)$, where the union is over $\mathcal{M}(f)$, is denoted by $\rho_{meas}(f)$.

M. Herman defines the rotation set of a homeomorphism precisely as $\rho_{meas}(f)$ (his notation is different), and his conditions on f and the volume that it preserves are needed in order to assure that

$$\rho_{meas}(f) \equiv \rho(\nu, f) \equiv \rho(\mu, f) \equiv \rho(x, f)$$

for every invariant measure ν , and for every point $x \in \mathbb{T}^d$, as is automatically the case in the circle (see [Her79]).

The condition imposed in thm. A can be written in the form²

$$\alpha \in \text{Conv}(\rho(f)) = \rho_{meas}(f)$$

When $d = 2$, it can be shown that $\rho(f)$ is convex, see [MZ91], so that $\rho(f) = \rho_{meas}(f)$.

3 Statement of the theorem

We can now restate our main theorem in a more precise way.

Theorem 3.1. *Let $d \in \mathbb{N}^*$, $\gamma > 0$ and $\tau > d$. Then, there exist $\varepsilon > 0$ and $s_0 > 0$ such that if $\alpha \in \mathbb{T}^d$ and $f \in \text{Diff}_0^\infty(\mathbb{T}^d)$ satisfy*

1. $\alpha \in DC(\gamma, \tau)$
2. $d_0(f(\cdot), R_\alpha) < \varepsilon$ and $d_{s_0}(f(\cdot), R_\alpha) < 1$
3. $\alpha \in \rho_{meas}(f)$

then f is C^∞ conjugate to R_α . Moreover, the conjugation can be chosen close to the Id .

Since such results tend to generalize to finite differentiability, we expect the following conjectural theorem to be true.

²By Conv we denote the convex hull of a set, i.e. the smallest closed convex set containing the given one.

Theorem 3.2 (Conjectural). *Let $d \in \mathbb{N}^*$, $\gamma > 0$ and $\tau > d$. Then, there exist $\varepsilon > 0$ and $\kappa, s_0 > 0$ with $0 < \kappa < s_0$ such that, if α and $f \in \text{Diff}^s(\mathbb{T}^d)$ with $s > s_0$ satisfy the conditions of theorem 3.1 in items 1 – 3, then f is $C^{s-\kappa}$ conjugate to R_α . The conjugation can be chosen close to the Id .*

This conjectural theorem is implied, for instance, by the proof in [Her79], which is carried out by approximation of finitely differentiable mappings by analytic ones. The proof is valid as we point out in the answer to the following question.

Question 3.1. *Does thm 3.1 hold true in the real analytic category?*

The is of course yes and an easy but non-optimal argument is as follows. If the perturbation is small enough in some analytic norm, then it is small enough in C^∞ . Therefore, thm 3.1 applies and the diffeomorphism is C^∞ conjugate to the rigid rotation R_α . As a consequence, its rotation set is reduced to $\{\alpha\}$, and M. Herman's proof can be applied by just dropping the volume preservation assumption.

4 Proof of theorem 3.1

The proof relies on two lemmas. The first one is a K.A.M. lemma of very classical flavour and estimates, and represents one step of the K.A.M. scheme that constructs successive conjugations reducing the diffeomorphism f to the rigid rotation R_α . The second lemma is a geometric one and relates the size of the perturbation with ρ_{meas} .

For the scheme to produce a converging product of conjugations, two conditions are needed. The first, more standard one, is a closeness to a rotation condition in an appropriate topology, and its general form is like the one of item 2 of the statement of thm 3.1. The second one is used in making sure that the perturbed diffeomorphism f does not drift away from R_α : clearly, if β is a vector with rational coordinates, very close to α , the two corresponding rotations are not conjugate.³ Such a condition can be imposed on the rotation set of f , the perturbed diffeomorphism, and a possible condition would then be

$$\rho(f) \equiv \{\alpha\}$$

where $\rho(f)$ is defined in paragraph 2.2. In fact, the following weaker condition would be sufficient:

$$\exists x \in \mathbb{T}^d, \rho(x, f) = \{\alpha\}$$

However, it is enough that $\alpha \in \text{Conv}\rho(f) = \rho_{meas}(f)$, which is exactly condition in item 3 of the theorem.

All three conditions are weaker than

$$\exists \Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d, \|\Phi - Id\|_{L^\infty} < \infty$$

³ We remark that the rotation set of a diffeomorphism of \mathbb{T}^d is only preserved by conjugations that are isotopic to the Id .

a measurable mapping, such that

$$\Phi \circ F = R_\alpha \circ \Phi$$

This last condition implies that for *a.e.* $x \in \mathbb{T}^d$

$$\rho(x, f) = \{\alpha\}$$

If Φ is assumed to be continuous, this holds for every $x \in \mathbb{T}^d$ and in fact $\rho(f) = \{\alpha\}$.

Finally, this last condition is weaker than the existence of $\phi : \mathbb{T}^d \rightarrow \mathbb{T}^d$, surjective and respectively measurable or continuous, such that

$$\phi \circ f = \mathbb{R}_\alpha \circ \phi$$

In the one-dimensional case, the existence of such a semi-conjugation is automatic as soon as the rotation number of the homeomorphism is irrational. In the higher-dimesnional case, however, the existence of a semi-conjugation to a minimal rotation is an additional and restrictive hypothesis, and brings us back to the context of the conjecture mentioned in the introduction.

4.1 Inductive lemma

We now state and recall the proof of the inductive conjugation lemma.

Lemma 4.1. *Let $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d$ and $f \in \text{Diff}^\infty(\mathbb{T}^d)$, and call $\|f - R_\alpha\|_{C^s} = \varepsilon_s$. Then, for some absolute constant $C > 0$ and for every $N \in \mathbb{N}^*$ such that*

$$C\gamma N^{2\tau+d+2}\varepsilon_0 < 1$$

there exists $\phi \in \text{Diff}^\infty(\mathbb{T}^d)$ such that

$$\phi \circ f \circ \phi^{-1} = f'$$

and the following hold true for the diffeomorphism $f' \in \text{Diff}^\infty$ thus defined.

There exists $\beta \in \mathbb{T}^d$ such that $\|f' - R_\beta\|_{C^s} = \varepsilon'_s$ satisfies

$$\varepsilon'_s \leq C_{s,s'} \left(N^{s+2\tau+d+2}\varepsilon_0^2 + N^{\tau+d/2}\varepsilon_0\varepsilon_s + N^{s-s'+d}(1 + N^{s+\tau+d/2}\varepsilon_0)\varepsilon_{s'} \right)$$

for every $0 \leq s \leq s' < \infty$.

Moreover, the conjugation ϕ satisfies

$$\|\phi\|_s \leq C_s \gamma N^{s+\tau+d/2}\varepsilon_0$$

Naturally, $\beta \simeq \alpha + \int (f - R_\alpha) d\mu$ and it represents a drift of the perturbed diffeomorphism with respect to R_α . The constants appearing in the statement depend on τ and d , but not on N . The proof is classical, but we sketch it for the sake of completeness.

Proof. Let $d = 2$ in order to simplify notation, without any loss of generality. Then,

$$f(\cdot) = R_\alpha + \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix}$$

where $f_i(\cdot) : \mathbb{T}^2 \rightarrow \mathbb{R}$ for $i = 1, 2$, are small in the C^∞ topology.

If we call

$$\phi(\cdot) = Id + \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix}$$

where $\phi_i(\cdot) : \mathbb{T}^2 \rightarrow \mathbb{R}$ for $i = 1, 2$, then, for the conclusion of the lemma to be true, they need only satisfy the equation

$$\phi_i(\cdot) \circ R_\alpha - \phi_i(\cdot) + \dot{T}_N f_i(\cdot) = 0$$

Such functions ϕ_i exist and are uniquely defined in $C_0^\infty(\mathbb{T}^2)$. They satisfy the estimate

$$\|\phi_i(\cdot)\|_s \leq C_s \gamma N^{s+\tau+d/2} \|f_i\|_0$$

Then, we can calculate

$$\begin{aligned} \phi \circ f \circ \phi^{-1} &= \left(Id + \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix} \right) \circ \left(R_\alpha + \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} \right) \circ \left(Id - \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix} + O(\phi_i^2) \right) \\ &= \left(Id + \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix} \right) \circ \left(R_\alpha - \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix} + T_N \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} + O(\cdot) \right) \\ &= R_\alpha + \begin{pmatrix} \hat{f}_1(0) \\ \hat{f}_2(0) \end{pmatrix} + \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix} \circ R_\alpha - \begin{pmatrix} \phi_1(\cdot) \\ \phi_2(\cdot) \end{pmatrix} + \dot{T}_N \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} + O(\cdot) \\ &= R_\alpha + \begin{pmatrix} \hat{f}_1(0) \\ \hat{f}_2(0) \end{pmatrix} + O(\phi_i^2, \partial(\dot{T}_N f_i) \cdot \phi_i, \partial(\phi_i) \cdot \phi_i, R_N f_i \circ (Id - \phi_i)) \end{aligned}$$

The $O(\cdot)$ term in the last line, which, anticipating the next section we call

$$\begin{pmatrix} f'_1(\cdot) \\ f'_2(\cdot) \end{pmatrix}$$

so that

$$\phi \circ f \circ \phi^{-1} = R_\alpha + \begin{pmatrix} \hat{f}_1(0) \\ \hat{f}_2(0) \end{pmatrix} + \begin{pmatrix} f'_1(\cdot) \\ f'_2(\cdot) \end{pmatrix}$$

can be estimated in the C^s -norm by

$$C_{s,s'} \left(N^{s+2\tau+d+2} \varepsilon_0^2 + N^{\tau+d/2} \varepsilon_0 \varepsilon_s + N^{s-s'+d} (1 + N^{s+\tau+d/2} \varepsilon_0) \varepsilon_{s'} \right)$$

This concludes the proof of the lemma. \square

4.2 A posteriori estimate on the obstruction

The following elementary and well known observation establishes a relation between the displacement of points in the torus \mathbb{T}^d under a diffeomorphism g with its rotation set $\rho(g)$.

Lemma 4.2. *Let $g \in \text{Diff}_0^\infty(\mathbb{T}^d)$ and $\beta \in \mathbb{T}^d$. If there exists $x \in \mathbb{T}^d$ such that $\rho(x, g) = \{\beta\}$, then $\beta \in \text{Conv}\{G(\tilde{x}) - \tilde{x}, x \in \mathbb{T}^d\}$.*

Inspection of the proof shows that the condition on the existence of an orbit rotating at speed β can be relaxed to $\beta \in \rho(g)$.

Proof. Let $x \in \mathbb{T}^d$ be such that $\rho(x, g) = \{\beta\}$. Then,

$$\frac{G^n(\tilde{x}) - \tilde{x}}{n} = \frac{G(G^{n-1}(\tilde{x})) - G^{n-1}(\tilde{x}) + G(G^{n-2}(\tilde{x})) - G^{n-2}(\tilde{x}) + \cdots + G(\tilde{x}) - \tilde{x}}{n}$$

converges to β . Since the right-hand side is an element of $\text{Conv}\{G(\tilde{x}) - \tilde{x}, x \in \mathbb{T}^d\}$ and the latter set is closed, the lemma is proved. \square

In the context of lemma 4.1, we obtain the following corollary.

Corollary 4.3. *There exists an absolute constant $C > 0$ depending only on d such that, under the hypotheses of lem. 4.1, and assuming additionally that $\alpha \in \text{Conv}(\rho(f))$,*

$$\left\| \begin{pmatrix} \hat{f}_1(0) \\ \hat{f}_2(0) \end{pmatrix} \mod \mathbb{Z}^2 \right\| \leq C\varepsilon'_0$$

In the proof we assume for simplicity that $\rho(x, f) = \{\alpha\}$ for some $x \in \mathbb{T}^2$. The proof of the corollary as it is stated follows easily.

Proof. If $\rho(x, f) = \{\alpha\}$, then $\rho(\varphi(x), f') = \{\alpha\}$. Then, by lemma 4.2,

$$\alpha \in \text{Conv}(F'(\cdot) - Id)$$

Consequently,

$$0 \in \text{Conv}(F'(\cdot) - \alpha) = \text{Conv} \left(\begin{pmatrix} \hat{f}_1(0) \\ \hat{f}_2(0) \end{pmatrix} + \begin{pmatrix} f'_1(\cdot) \\ f'_2(\cdot) \end{pmatrix} \right)$$

The corollary follows directly. \square

Since invariant measures are accumulation points of Dirac measures uniformly distributed on finite segments of orbits, we can relax the condition $\alpha \in \rho(f)$ to $\alpha \in \rho_{\text{meas}}(f)$: We must have

$$0 = \int_{\mathbb{T}^d} \left(\begin{pmatrix} \hat{f}_1(0) \\ \hat{f}_2(0) \end{pmatrix} + \begin{pmatrix} f'_1(\cdot) \\ f'_2(\cdot) \end{pmatrix} \right) d(\phi_* \nu)$$

for every (fixed) $\nu \in \mathcal{M}(f)$ such that $\rho(\nu, f) = \alpha$. We immediately get the same estimate as in cor. 4.3.

4.3 KAM scheme and convergence

The estimates provided by lemma 4.1 are sufficient for the convergence of the corresponding scheme, provided that some smallness conditions are satisfied, and that we can linearize around the same rotation α throughout the scheme, so that no "counter-term" is needed (as in the normal form version of the theorem in [Her79]), and the Diophantine condition can be kept constant throughout the scheme. This second condition is assured by the existence of an orbit rotating like R_α , and by corollary 4.3.

Let us state this formally in the following proposition.

Proposition 4.4. *Let $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d$ and $f = f_1 \in \text{Diff}^\infty(\mathbb{T}^d)$, and call $\|f_1 - R_\alpha\|_{C^s} = \varepsilon_{s,1}$. Then, there exist $\epsilon > 0$ and $s_0 \in \mathbb{N}^*$ such that if*

$$\varepsilon_{0,1} < \epsilon \text{ and } \varepsilon_{s_0,1} < 1$$

and if

$$\alpha \in \rho(f_1)$$

then there exist inductively defined sequences $\phi_n \in \text{Diff}_0^\infty(\mathbb{T}^d)$ and $f_n \in \text{Diff}_0^\infty(\mathbb{T}^d)$ such that

$$\phi_n \circ f_n \circ \phi_n^{-1} = f_{n+1}$$

with

$$\|f_n - R_\alpha\|_{C^s} = \varepsilon_{s,n} \xrightarrow{n \rightarrow \infty} 0, \forall s \in \mathbb{N} \quad (3)$$

Moreover,

$$\prod_{k=1}^n \phi_k \xrightarrow{n \rightarrow \infty} \phi \in \text{Diff}^\infty(\mathbb{T}^d)$$

Clearly, this proposition implies thm. 3.1. The proposition is proved by iteratively applying lemma 4.1 and then corollary 4.3 in the following, now classical, way.

Proof. Let $N = N_1 \in \mathbb{N}^*$ be large enough, chose $\sigma > 0$, and define inductively

$$N_n = N_{n-1}^{1+\sigma} = N^{(1+\sigma)^{n-1}}$$

to be the order of truncation at the n -th step, as in the proof of lem. 4.1.

Assume, now, that ϕ_{n-1} has already been constructed, so that f_n is well defined. Suppose, additionally, that f_n satisfies the hypotheses of lem. 4.1 for $\varepsilon_s = \varepsilon_{s,n}$ and $N = N_n$. Then, application of lem. 4.1 and then of cor. 4.3 grants the existence of ϕ_n such that

$$f_{n+1} = \phi_n \circ f_n \circ \phi_n^{-1}$$

satisfies, for all $0 \leq s \leq s' < \infty$, the inequality

$$\begin{aligned} \varepsilon_{s,n+1} \leq C_{s,s'} \left(N_n^{s+2\tau+d+2} \varepsilon_{0,n}^2 + N_n^{\tau+d/2} \varepsilon_{0,n} \varepsilon_{s,n} \right. \\ \left. + N_n^{s-s'+d} (1 + N_n^{s+\tau+d/2} \varepsilon_{0,n}) \varepsilon_{s',n} \right) \end{aligned} \quad (4)$$

We remind that $\varepsilon_{n,s}$, defined after eq. 3, represents the distance of the diffeomorphism f_n from the *fixed* rotation R_α , thanks to the a posteriori estimate of cor. 4.3.

Since the term in $\varepsilon_{0,n}\varepsilon_{s',n}$ is not present in thm. 10 of [FK09], we partially reprove the convergence of the scheme. In fact, only the two main steps, lemmata 11 and 14 of the reference, have to be proved for the kind of estimates that we have herein⁴. We therefore need the following lemma, which is proved in the appendix, section 6.

Lemma 4.5. *Let $\varepsilon_{s,n}$ satisfy the inductive estimates of eq. 4. If, moreover*

$$\begin{aligned}\varepsilon_{0,1} &< N_1^{-\gamma_0} \\ \varepsilon_{s_0,1} &< N_1^b\end{aligned}$$

for some appropriately chosen $\gamma_0, b > 0$, then the double sequence $\varepsilon_{s,n}$ is well defined and for all n

$$\begin{aligned}\varepsilon_{0,n} &< N_n^{-\gamma_0} \\ \varepsilon_{s_0,n} &< N_n^b\end{aligned}$$

We note, en passant, that this lemma implies that, under the relevant smallness conditions on $\varepsilon_{s,0}$, the smallness conditions of lemma 4.1 are satisfied for all n . Therefore, the double sequence $\varepsilon_{s,n}$ is well defined and we only need to establish its convergence.

We then show that, given the decay and growth rates granted by the previous lemma, we can actually do slightly better. The following lemma allows us to bootstrap exactly like in [FK09] and conclude the convergence.

Lemma 4.6. *Let $\varepsilon_{s,n}$ satisfy the inductive estimates of eq. 4. If, moreover*

$$\begin{aligned}\varepsilon_{0,n} &= O(N_n^{-\gamma_0}) \\ \varepsilon_{s_0,n} &= O(N_n^b)\end{aligned}$$

with γ_0, b and s_0 as in the previous lemma then, there exist $\omega_0, \omega > 0$ such that

$$\begin{aligned}\varepsilon_{0,n} &= O(N_n^{-(1+\omega_0)\gamma_0}) \\ \varepsilon_{(1+\omega)s_0,n} &= O(N_n^b)\end{aligned}$$

Thus, we have successively proved the following assertions:

- at each step, the inductive smallness hypothesis for lem. 4.1 to be applicable is satisfied by f_n and N_n . Therefore, the double sequence $\varepsilon_{s,n}$ is well defined for all $n \in \mathbb{N}^*$ and $s \in \mathbb{N}$.

⁴ The discrepancy is due to the fact that in the reference the dynamical system considered is a cocycle, and the last term herein comes from composition of mappings, whereas in the context of cocycles mappings are composed uniquely with the exp and only products need to be considered.

- for every fixed $s \in \mathbb{N}$ and $\lambda \geq 0$

$$N_n^\lambda \varepsilon_{s,n} \xrightarrow{n \rightarrow \infty} 0$$

The shorthand $f_n - R_\alpha = O_{C^\infty}(N_n^{-\infty})$ is common. This is obtained by the fast convergence of $\varepsilon_{0,n}$ to 0, faster than any power of N_n , and the slow growth of $\varepsilon_{s,n}$ (as a fixed power of N_n) for every s fixed. Convexity estimates allow us to conclude that for every $0 < s' < s$, $\varepsilon_{s',n} \rightarrow 0$.

- the fast convergence of $\varepsilon_{s,n}$ to 0 and the fact that

$$\|\Phi_n - Id\|_s \leq C_s \gamma N_n^{s+\tau+d/2} \varepsilon_{0,n}$$

imply, together with eq. 1 that the product of successive conjugations

$$\prod_{k=1}^n \phi_k \in \text{Diff}^\infty(\mathbb{T}^d)$$

converges in the C^∞ topology to a well defined diffeomorphism ϕ .

This concludes the proof of the proposition, and thus of thm 3.1. \square

5 A remark on the proof

In the one-dimensional case, the theory of the rotation number for an orientation preserving homeomorphism of \mathbb{T}^1 is considerably stronger, thanks to the existence of a natural cyclic order (or of a total order in the covering space \mathbb{R}^1). The analogue of our argument in the one-dimensional case would be the following. Consider a Diophantine rotation α and perturb it. Suppose that one orbit with rotation number α survives under perturbation. Solve the linear cohomological equation and observe that if the obstruction (the mean value of the perturbation) is not of "second order", a contradiction would be established, e.g. by fitting a rational number between

$$\text{Conv}\{F(\tilde{x}) - \tilde{x}, x \in \mathbb{T}^1\} = \{F(\tilde{x}) - \tilde{x}, x \in \mathbb{T}^1\}$$

and α , see [KH96].

Of course, and this is a particularity of the one-dimensional theory, if one orbit has rotation number α , then all orbits do. This, however, is not an essential part of the proof of the existence of a smooth conjugacy to the rigid rotation, since the proof of the uniqueness of the rotation number is formally independent of the construction of the K.A.M. scheme and of the proof of its convergence. M. Herman in his thesis, [Her79], defines the rotation number for circle diffeomorphisms using an invariant measure instead of a combinatorial definition as in [KH96] or [dMvS93]. Accordingly to the one-dimensional case, his definition of the rotation set for diffeomorphisms preserving a volume form assures that the rotation set is thus reduced to a point, and this hypothesis is, in fact, needlessly strong.

6 Appendix

We now provide the missing proofs of lemmata 4.5 and 4.6. We note that, when eq. 4 is compared with the corresponding eq. 7.2 of [FK09], which in our notation reads

$$\varepsilon_{s,n+1} \leq C_{s,s'} \left(N_n^{a+Ms} \varepsilon_{0,n}^{1+\sigma_0} + N_n^{a+ms} \varepsilon_{0,n} \varepsilon_{s,n} + N_n^{a-(s'-s)\mu} (\varepsilon_{s',n} + N_n^{\bar{\mu}s'} \varepsilon_{0,n}) \right) \quad (5)$$

the agreeing terms correspond to the admissible choice of parameters (in the notation of the reference)

$$\begin{aligned} \sigma_0 &= 1 \quad , \quad a = 2\tau + d + 2 \\ M &= 1 \quad , \quad m = 0 \\ \mu &= 2 \quad , \quad \bar{\mu} = 0 \end{aligned}$$

However, there do not seem to exist admissible values of the parameters g , σ and κ of the reference for which either our estimates can be brought to the form of those of the reference, or for which the proof found therein produces convergence of the scheme for our type of estimates. Consequently, we take up the proof of convergence for our type of estimates and we remark that the broader values of parameters (or the broader scope of types of estimates) are obtained thanks to the fact that we consider a not as an "affine" parameter, but as a "homogeneous" one, when compared to γ_0 , b and s_0 .

The additional term appearing in eq. 4, namely $N_n^{2s-s'+a} \varepsilon_{0,n} \varepsilon_{s',n}$, is due to the composition of the conjugation with the perturbation (and is not present in the context of cocycles). By defining γ_0 , b and s_0 as multiples of a we manage to absorb the additional growth by N_n^s into $\varepsilon_{0,n}$, which is shown to decay fast enough.

Let us now proceed to the actual proofs of the lemmata.

Proof of lem. 4.5. In view of the estimate of eq. 4 by setting $s = 0$, $s' = s_0$ and $s = s' = s_0$, we only need to show that if $\varepsilon_{0,n} < N_n^{-\gamma_0}$ and $\varepsilon_{s_0,n} < N_n^b$, then

$$\varepsilon_{0,n+1} \leq C_{0,s_0} \left(N_n^a \varepsilon_{0,n}^2 + N_n^{a/2} \varepsilon_{0,n}^2 + N_n^{-s_0+a/2} (1 + N_n^{a/2} \varepsilon_{0,n}) \varepsilon_{s_0,n} \right)$$

and

$$\varepsilon_{s_0,n+1} \leq C_{s_0,s_0} \left(N_n^{s_0+a} \varepsilon_{0,n}^2 + N_n^{a/2} \varepsilon_{0,n} \varepsilon_{s_0,n} + N_n^{a/2} (1 + N_n^{s_0+a/2} \varepsilon_{0,n}) \varepsilon_{s_0,n} \right)$$

satisfy $\varepsilon_{0,n+1} < N_{n+1}^{-\gamma_0}$ and $\varepsilon_{s_0,n+1} < N_{n+1}^b$.

The inequalities that need to be verified in the limit of N_1 large enough read

$$\begin{aligned} a - 2\gamma_0 &< -(1 + \sigma)\gamma_0 \quad , \quad a/2 - 2\gamma_0 < -(1 + \sigma)\gamma_0 \\ -s_0 + a/2 + b &< -(1 + \sigma)\gamma_0 \quad , \quad -s_0 + a - \gamma_0 + b < -(1 + \sigma)\gamma_0 \\ s_0 + a - 2\gamma_0 &< (1 + \sigma)b \quad , \quad a/2 - \gamma_0 + b < (1 + \sigma)b \\ a/2 + b &< (1 + \sigma)b \quad , \quad s_0 + a - \gamma_0 + b < (1 + \sigma)b \end{aligned}$$

Substitution by $\gamma_0 = \lambda a$, $s_0 = \mu a$ and $b = \nu a$ where λ, μ, ν are positive reals, gives

$$(1 - \sigma)\lambda > 1 \quad (6a)$$

$$\mu - (1 + \sigma)\lambda - \nu > 1/2 \quad (6b)$$

$$\mu - \nu - \sigma\lambda > 1 \quad (6c)$$

$$2\lambda + (1 + \sigma)\nu - \mu > 1 \quad (6d)$$

$$\sigma\nu + \lambda > 1 \quad (6e)$$

$$\sigma\nu > 1/2 \quad (6f)$$

$$\mu - \sigma\nu - \lambda > 1 \quad (6g)$$

Inequality 6a implies that $\lambda > 1$, so that ineq. 6e is redundant. For ineq. 6b, 6c, 6d and 6g to be compatible, we need

$$(1 + \sigma)\lambda + \nu + 1/2 < 2\lambda + (1 + \sigma)\nu - 1$$

$$1 + \sigma\lambda + \nu < 2\lambda + (1 + \sigma)\nu - 1$$

$$1 + \sigma\nu + \lambda < 2\lambda + (1 + \sigma)\nu - 1$$

or equivalently

$$3/2 < (1 - \sigma)\lambda + \sigma\nu$$

$$2 < (2 - \sigma)\lambda + \sigma\nu$$

$$2 < \lambda + \nu$$

The first two ineq. are implied by 6a, 6e and 6f. We thus need to impose

$$1 > \sigma \quad (7a)$$

$$\lambda + \nu > 2 \quad (7b)$$

$$(1 - \sigma)\lambda > 1 \quad (7c)$$

$$\sigma\nu > 1/2 \quad (7d)$$

and choose μ such that

$$\max\{(1 + \sigma)\lambda + \nu + 1/2, 1 + \sigma\lambda + \nu, 1 + \sigma\nu + \lambda\} < \mu < 2\lambda + (1 + \sigma)\nu - 1$$

The conditions are equivalent to

$$\frac{\lambda - 1}{\lambda} > \sigma > \frac{1}{2\nu}$$

$$\lambda > \frac{2\nu}{2\nu - 1}$$

$$\nu > 1/2$$

□

At this point, we encourage the reader to check lemmata 12 and 13 in [FK09], as they prepare the following proof of the lemma corresponding to lem. 14 of the reference.

Proof of lem. 4.6. Let $\varepsilon_{0,n} < \bar{C}N_n^{-\gamma_0}$ and $\varepsilon_{s_0,n} < \bar{C}N_n^b$. We first prove that there exists ω_0 such that

$$\varepsilon_{0,n} = O(N_n^{-(1+\omega_0)\gamma_0})$$

We calculate directly

$$\begin{aligned} \varepsilon_{0,n+1} &\leq \bar{C}C_{0,s_0} \left(N_n^{a-\gamma_0} \varepsilon_{0,n} + N_n^{-s_0+a/2+b} \right) \\ &\leq \bar{C}C_{0,s_0}^2 \left(N_n^{a-\gamma_0} \left(N_{n-1}^{a-\gamma_0} \varepsilon_{0,n-1} + N_{n-1}^{-s_0+a/2+b} \right) + N_n^{-s_0+a/2+b} \right) \\ &\leq (\bar{C}C_{0,s_0})^2 \left(N_n^{a-\gamma_0} \left(N_{n-1}^{a-2\gamma_0} + N_{n-1}^{-s_0+a/2+b} \right) + N_n^{-s_0+a/2+b} \right) \\ &\leq (\bar{C}C_{0,s_0})^2 \left(N_n^{\frac{2+\sigma}{1+\sigma}a - \frac{3+\sigma}{1+\sigma}\gamma_0} + N_n^{\frac{3+2\sigma}{2+2\sigma}a - \gamma_0 - \frac{s_0-b}{1+\sigma}} + N_n^{-s_0+a/2+b} \right) \end{aligned}$$

Again in the limit $n \rightarrow \infty$ we only need to verify that

$$\begin{aligned} \frac{2+\sigma}{1+\sigma}a - \frac{3+\sigma}{1+\sigma}\gamma_0 &< -(1+\sigma)(1+\omega_0)\gamma_0 \\ \frac{3+2\sigma}{2+2\sigma}a - \gamma_0 - \frac{s_0-b}{1+\sigma} &< -(1+\sigma)(1+\omega_0)\gamma_0 \\ -s_0 + a/2 + b &< -(1+\sigma)(1+\omega_0)\gamma_0 \end{aligned}$$

The first inequality holds true for

$$0 < \omega_0 < \frac{2\lambda - \sigma - 2 - \lambda\sigma(1+\sigma)}{\lambda(1+\sigma)^2}$$

The second holds true as long as

$$\mu > \frac{3}{2} + \sigma + \nu + (1+\sigma)((1+\sigma)(1+\omega_0) - 1)\lambda$$

This inequality is verified for $\omega_0 = 0$, by 7a and 7c and by the choice of μ , and therefore also verified for ω_0 small enough.

The third inequality is equivalent to

$$\mu > \frac{1}{2} + \nu + (1+\sigma)(1+\omega_0)\lambda$$

which is verified provided that ω_0 is small enough, by the choice of μ .

The second assertion of the lemma follows directly from gain in the speed of convergence for $\varepsilon_{0,n}$ (i.e. from the fact that we can replace γ_0 by $(1+\omega_0)\gamma_0$ for some fixed $\omega_0 > 0$), and the fact that in the inequalities s_0 6b, 6c, 6d and 6g μ and λ appear with opposite signs, so that increase in μ can be compensated by the increase in $\lambda \rightarrow (1+\omega_0)\lambda$ without the inequality being violated for the same choice of ν . The eventual multiplicative constants can be absorbed in the exponents in the limit of n large enough. \square

This last lemma can be used in order to bootstrap and obtain prop. 4.4.

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